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Note

Edge-coloring critical graphs with high degree ☆

Lian-ying Miao[a,*], Jian-liang Wu[b]

[a] School of Science, Shandong Agricultural University, 271018 Taian, Shandong, China

[b] Shandong University of Science and Technology, Jinan, 250031, China

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Abstract

It is proved here that any edge-coloring critical graph of order n and maximum degree $\Delta \geqslant 8$ has the size at least $3(n+\Delta-8)$. It generalizes a result of Hugh Hind and Yue Zhao.

Keywords: Edge coloring; Edge chromatic number; Critical graph

All graphs considered are simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If $v \in V(G)$, then its neighbor $N_G(v)$ and its degree $d(v)$ are the set and number of the vertices in G adjacent to v, respectively. We denote the maximum (minimum) degree of G by $\Delta(G)(\delta(G))$. A vertex v of G is *major* if $d(v)=\Delta(G)$. Let n_j be the number of vertices of degree j in G. We use $1^{n_1}2^{n_2}\ldots\Delta^{n_\Delta}$ to denote the *degree-list* of G. Note that if $n_j=0$, then the factor j^{n_j} is omitted in G. The *edge chromatic number* $\chi'(G)$ is the minimum number of colors needed to color the edges of G in such a way that no two adjacent edges are assigned the same color. Vizing's theorem states that for any graph G, $\chi'(G)=\Delta$ or $\Delta+1$. A graph G is said to be of *class one* if $\chi'(G)=\Delta$, and of *class two* if $\chi'(G)=\Delta+1$. A graph G is *critical* if G is connected, and of class two and $\chi'(G-e)<\chi'(G)$ for any edge e of G. If a graph G is critical and G has maximum degree Δ, we say that G is *Δ-critical*.

Vizing [4] conjectured that every Δ-critical graph of order n has at least $(n\Delta-n+3)/2$ edges. Some results about the conjecture were obtained, seeing [1,3,6,7]. Here we use the idea of Yap in [6] to prove that if G is a Δ-critical graph of order n and $\Delta \geqslant 8$, then G has at least $3(n+\Delta-8)$ edges. It generalizes a result of Hugh Hind and Yue Zhao

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* Corresponding author.

E-mail addresses: lianying@sdau.edu.cn (L.-y. Miao), j.l.wu@china.com (J.-l. Wu).

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in [2]: any graph of maximum degree $\geqslant 8(\geqslant 9)$ that can be embedded in a surface of characteristic $\geqslant 0(\geqslant -1$, respectively) is of class one, since every connected graph G of order n which can be embedded in a surface of characteristic ε has at most $3(n-\varepsilon)$ edges.

To prove our result, we will make use of the famous lemma, which can be found in many books.

Lemma 1 (Vizing's Adjacency Lemma). *Let G be a Δ-critical graph and let v, w be adjacent vertices of G with $d(v)=k$. Then*

(a) *if $k<\Delta$, then w is adjacent to at least $\Delta - k + 1$ major vertices*;
(b) *if $k=\Delta$, then w is adjacent to at least two major vertices*;
(c) *G contains at least $\Delta - \delta(G) + 2$ major vertices.*

Theorem 2. *If G is a critical graph of order n and maximum degree Δ, where $\Delta \geqslant 8$, then $|E(G)| > 3(n+\Delta-8)$.*

Proof. Assume, to the contrary, that there exists a critical graphs G of order n and maximum degree Δ, where $\Delta \geqslant 8$, such that $|E(G)| \leqslant 3(n+\Delta-8)$. Then we have

$$2|E(G)| = 2n_2 + 3n_3 + \cdots + \Delta n_\Delta \leqslant 6(n_2 + n_3 + \cdots + n_\Delta + \Delta - 8)$$

$$\sum_{i=7}^{\Delta}(i-6)n_i \leqslant 4n_2 + 3n_3 + 2n_4 + n_5 + 6(\Delta - 8) \tag{1}$$

Let $n_\Delta(i_2, i_3, \ldots, i_{\Delta-1})$ be vertices of degree Δ which have i_2 neighbors of degree 2, i_3 of degree $3, \ldots, i_{\Delta-1}$ of degree $\Delta-1$. Since G is Δ-critical, by counting the number of edges with one end of degree j and the other end of degree Δ, we have

$$2n_j \leqslant \sum i_j n_\Delta(i_2, i_3, \ldots, i_{\Delta-1})$$

for any $j = 2, 3, \ldots, \Delta-1$, where the summation is over all possible $i_2, i_3, \ldots, i_{\Delta-1}$. By the proof of Yap [6] (also referring to Theorem 7.1 in [7, p. 50]), the inequality was improved for $j=3,4$ as follows. For $j=3$, let r be the number of vertices of degree 3 each of which is adjacent to exactly one vertex of degree $\Delta-1$

$$3(n_3 - r) + 2r \leqslant \sum i_3 n_\Delta(i_2, i_3, \ldots, i_{\Delta-1}).$$

For $j=4$, let s be the number of vertices of degree 4 each of which is adjacent to exactly one vertex of degree $\Delta-2$, t the number of vertices of degree 4 each of which is adjacent to just one vertex of degree $\Delta-1$, and u the number of vertices of degree 4 each of which is adjacent to just two vertices of degree $\Delta-1$

$$4(n_4 - s - t - u) + 3s + 3t + 2u \leqslant \sum i_4 n_\Delta(i_2, i_3, \ldots, i_{\Delta-1}).$$

Thus, we have

$$2n_2 + \frac{3(n_3 - r) + 2r}{2} + \frac{4(n_4 - s - t - u) + 3s + 3t + 2u}{3} + \sum_{i=5}^{\Delta-1} \frac{2n_i}{i-1}$$

$$\leqslant \sum_{j=2}^{\Delta-1} \sum \frac{i_j n_\Delta(i_2, i_3, \ldots, i_{\Delta-1})}{j-1}$$

$$= \sum n_\Delta(i_2, \ldots, i_{\Delta-1}) \sum_{j=2}^{\Delta-1} \frac{i_j}{j-1}.$$

If $n_\Delta(i_2, \ldots, i_{\Delta-1}) \neq 0$, then $\sum_{j=2}^{\Delta-1} (i_j/(j-1)) \leqslant 1$. So we have

$$n_\Delta \geqslant 2n_2 + \frac{3n_3 - r}{2} + \frac{4n_4 - s - t - 2u}{3} + \sum_{i=5}^{\Delta-1} \frac{2n_i}{i-1} \tag{2}$$

and

$$(\Delta - 7)n_{\Delta-1} + 2n_\Delta \geqslant 4n_2 + 3n_3 + 2n_4 + n_5 + 2(n_4 - s - t)/3 + (n_{\Delta-1} - r - u)$$

$$+ \{n_7/3 - u/3 + (\Delta - 8)n_{\Delta-1}\} + \sum_{i=6}^{\Delta-1} n_i/\Delta$$

$$\geqslant 4n_2 + 3n_3 + 2n_4 + n_5 + \sum_{i=6}^{\Delta-1} n_i/\Delta.$$

So

$$\sum_{i=7}^{\Delta} (i-6)n_i \geqslant 4n_2 + 3n_3 + 2n_4 + n_5 + \sum_{i=6}^{\Delta-1} n_i/\Delta + (\Delta - 8)n_\Delta. \tag{3}$$

Combining (1) and (3), we have

$$6(\Delta - 8) \geqslant \sum_{i=6}^{\Delta-1} n_i/\Delta + (\Delta - 8)n_\Delta$$

that is,

$$\sum_{i=6}^{\Delta-1} n_i/\Delta + (\Delta - 8)(n_\Delta - 6) \leqslant 0. \tag{4}$$

Suppose that $\Delta = 8$. By (4), $n_6 = n_7 = 0$. At the same time, $r = s = t = u = 0$. Thus by (2), we have

$$2n_8 \geqslant 2(2n_2 + 3n_3/2 + 4n_4/3 + n_5/2) \geqslant 4n_2 + 3n_3 + 2n_4 + n_5 + 2n_4/3. \tag{5}$$

Combining (1) and (5), we have $n_4=0$. Now let us prove $n_5=0$. If $n_5\neq 0$, let n_5' be the number of vertices of degree 5 each of which is adjacent to 5 vertices of degree 8. Then

$$n_8\geqslant 2n_2+3n_3/2+[5n_5'+4(n_5-n_5')]/4=2n_2+3n_3/2+(4n_5+n_5')/4$$
$$>2n_2+3n_3/2+n_5/2.$$

It is a contradiction to (1). So $n_5=0$.

From the above, we know that $2n_8=4n_2+3n_3$. By Theorem 5 in [6], there is no critical graph with degree-list $2^{n_2}8^{n_8}$ and $2n_2=n_8$. So $n_3\neq 0$. By Lemma 2 in [2], there is no critical graph with degree-list $3^{n_3}8^{n_8}$ and $2n_8=3n_3$. So $n_2\neq 0$. Thus the degree-list of G must be $2^{n_2}3^{n_3}8^{n_8}$ and $2n_8=4n_2+3n_3$. By Lemma 2 in [5], it is impossible. Hence $\Delta\geqslant 9$.

Since $\Delta\geqslant 9$, it follows from (4) that $n_\Delta\leqslant 6$. By Lemma 1, $\delta(G)\geqslant\Delta-n_\Delta+2\geqslant\Delta-4\geqslant 5$. If $\Delta=9$ and $\delta(G)=5$, then $n_6=n_7=n_8=0$, that is, G has degree-list $5^{n_5}9^{n_9}$. Since $5n_5\leqslant 4n_9$, $n_9>n_5$. So $3n_9>n_5+6$, a contraction to (1). If $\Delta=9$ and $\delta\geqslant 6$, then

$$2|E(G)|\geqslant\Delta(\Delta-\delta+2)+\delta[n-(\Delta-\delta+2)]>6(n+3)$$

a contraction to (1), too.

So we have $\Delta\geqslant 10$ and $\delta(G)\geqslant 6$. If $\delta(G)=6$, then $n_\Delta=6$. It follows from (4) that $n_6=\cdots=n_{\Delta-1}=0$, that is impossible. If $\delta(G)>6$, it is easy to check that

$$2|E(G)|\geqslant\Delta(\Delta-\delta+2)+\delta[n-(\Delta-\delta+2)]>6(n+\Delta-8).$$

Thus we complete the proof. □

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References

[1] S. Fiorini, R.J. Wilson, Edge-Colorings of Graphs, Research Notes in Maths, Vol. 16, Pitman, London, 1977.
[2] H. Hind, Y. Zhao, Edge colorings of graphs embedable in a surface of low genus, Discrete Math. 190 (1998) 107–114.
[3] L.S. Mel'nikov, The chromatic class and the location of a graph on a closed surface, Mat. Zametki 7 (1970) 671–681; Math. Notes 7 (1970) 405–411.
[4] V.G. Vizing, Some unsolved problems in graph theory, Russian Math. Surveys 23 (1968) 125–142.
[5] Z.D. Yan, Y. Zhao, Edge colorings of embedded graphs, Graphs Combin. 16 (2000) 245–256.
[6] H.P. Yap, On graphs critical with respect to edge-colourings, Discrete Math. 37 (1981) 289–296.
[7] H.P. Yap, Some Topics in Graph Theory, Cambridge University Press, New York, 1986.